АЛГОРИТМИ

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OREDRING BY BINARY-POSITIONAL LOGICAL OPERATIONS

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The article shows that the operations of mathematical logic and algebraic methods of description of algorithms based on mathematical logic do not take into account the positions. New operations of α-, β-, αβ-conjunction (disjunction) and positional inverting have been defined, which take into account the positions. The properties of these operations have been formulated and proved. The mutual unambiguousness has been established between the classical operations of conjunction (disjunction) and the operations of α-, β-, αβ-conjunctions (-disjunctions). The ordering of formulas by positions and the possibility of performing identical transformations of the ordering have been proved.

Key words*: *α-conjunction, β-conjunction, αβ-conjunction, α-disjunction, β-disjunction, αβ-disjunction, positional inverting.

1. Introduction

A position of something is its location or placing. The term is used in mathematics, computer sciences, informatics and other areas. For example, mathematical induction [1] uses the initial value of a variable, and programming and information technology has an initial instruction [2], which, like any other programming instruction, has a unique number or name. The “initial” value, as well as the “numbers” of the instructions, are actually their positions. In algorithms, the position of operators has a key value. However, in classical mathematical logic [1], as well as, in particular, algebraic theories of algorithms, the positions are not taken into account. Particularly, this can be seen in the system of algorithmic algebras [3, 4], the modified system of algorithmic algebras [5, 6], the primitive programming algebra [7, 8], the algebra of algorithms [9], and the modified algebra of algorithms [10] and their applications [11, 12, 13].

The operations of classical mathematical logic [1], in particular conjunction (&), disjunction (□) do not take into account the positions of logical constants, variables and predicates. This also applies to non-classical mathematical logic, in particular the propositional modal logic, the propositional dynamic logic, the linear propositional temporal logic, the multivalued logic, the fuzzy logic, the intuitionistic logic, and so on.

The system of algorithmic algebras [3] and the modified system of algorithmic algebras [5] are formed by two algebras – the algebra of logic extended for three-digit alphabet and the operator algebra. The signature of the operator algebra contains the operations of the composition, alternatives, asynchronous disjunction, filtration, cycle, and synchronization [3, 5]. But in the operator algebra, as well as in the algebra of logic extended for three-digit alphabet, there is no means for taking into account the operator positions.

The signature of the primitive programming algebras [7, 8] contains a substitution (superposition), branching and parameterized cycling that use the concept of tuple. However, the signature operations of the primitive programming algebras also do not take into account the positions.

The signature of the algebra of algorithms [9] is formed by the operations of sequencing, elimination, paralleling, cyclic sequencing, elimination and paralleling. In the modified algebra of algorithms, the signature of the algebra of algorithms is complemented by the operation of multi-elimination [10]. These algebras use a complex system of two-dimensional operation symbols. But the operations of these algebras also do not take into account the positions of constants, variables and operators over which they are performed in the explicit form.

It is known that for performing the ordering, the relation of partial (□) or strict order (<) [14] is used. There exists an axiom of complete ordering and an axiom of choice. However, the relation □ and < also do not use positions. In addition, their application does not allow the implementation of identical transformations of orderings.

Particularly, it is important to take into consideration the positions in mathematical logic, as the basis of modern mathematics. In addition, for an adequate description of algorithms, and not only algorithms, it is necessary to take into account the positions of operators that form the algorithms.

This paper is dedicated to solving the problem of defining the logical operations, which would take into account the positions.

2. Unary- and Binary-Positioning of Relations and Operations

For example, the set $C = A \times B$ consists of the ordered pairs ($C = \{(a, b) \mid a \in A, b \in B\}$). If $C = \{(a, a) \mid a \in A\}$ and $A = \{0, 1\}$, then $C = \{(a, a) \mid a \in A\} = \{(0,0), (1, 1)\}$.

Elements $a \in A$ and $b \in B$ of the ordered pair *(a, b)* have well-defined positions, but they are not explicitly specified in the ordered pair (a, b).

For explicit assigning the positions of the elements $a \in A$ and $b \in B$ of the ordered pair *(a, b)* we use the letters of the Greek alphabet α and β. The positions of elements a and b of the ordered pair *(a, b)* are written in the form of right lower indices. In this case, the ordered pair *(a, b)* will look like (a_α, b_β), in which $a \in A$ is in the position α and $b \in B$ is in the position β, which can be written in the simplified form as $a_\alpha \in A$ and $b_\beta \in B$. In this case $C = A \times B = \{(a_\alpha, b_\beta) \mid a_\alpha \in A, b_\beta \in B\}$.

For example, if $A = \{0, 1\}$ and $B = \{0, 1\}$, then $C = \{(a_\alpha, b_\beta) \mid a_\alpha \in A, b_\beta \in B\} = \{(0_\alpha,0_\beta), (0_\alpha, 1_\beta), (1_\alpha, 0_\beta), (1_\alpha, 1_\beta)\}$. If $C = A \times A = \{(a_\alpha, a_\beta) \mid a_\alpha \in A, a_\beta \in A \}$, then for $A = \{0, 1\}$ we have $C = \{(a_\alpha, a_\beta) \mid a_\alpha \in A, a_\beta \in A\} = \{(0_\alpha,0_\beta), (0_\alpha, 1_\beta), (1_\alpha, 0_\beta), (1_\alpha, 1_\beta)\}$, but not $C = \{(a, a) \mid a \in A\} = \{(0,0), (1, 1)\}$, as it happens in the case without explicit assigning of the elements positions.

Definition 1. The places α and β of the location of the elements $a \in A$ and $b \in B$ of the ordered pair (a, b) of the Cartesian product $A \times B$ in the sets A and B, are called *elementary positions of elements*.

Definition 2. The ordered pair (a_α, b_β) with explicit assigning of the elementary positions of the elements $a_\alpha \in A$ and $b_\beta \in B$ of the Cartesian product $A \times B$ in the sets A and B, is called a *positional ordered pair*, and the elements with elementary positions are called *positional elements*.

Definition 3. The positional ordered pair (a_α, b_β) of the Cartesian product $A \times B$ of the sets A and B with two-positional elements $a_\alpha \in A$ and $b_\beta \in B$, is called *a binary-positional ordered pair* and the positional ordered pairs (a_α, b_α) and (a_β, b_β) with one-positional elements $a_\alpha, a_\beta \in A$ and $b_\alpha, b_\beta \in B$ are called *unary-positional ordered pairs*.

Both positional elements of the unary-positional ordered pair have one and the same position. For example, the positional elements of the unary-positional ordered pair (c_α, d_α) have the elementary position α, and the positional elements of the unary-positional ordered pair (g_β, h_β) have the elementary position β.

Definition 4. A random set $C = A \times B = \{(a_\alpha, b_\beta) \mid a_\alpha \in A, b_\beta \in B\}$, which is formed by the binary-positional ordered pairs (a_α, b_β), is called a *binary-positional binary relation*, set or defined on the sets A and B. For $C = A \times B = \{(a_\alpha, b_\alpha) \mid a_\alpha \in A, b_\alpha \in B\}$ and $C = A \times B = \{(a_\beta, b_\beta) \mid a_\beta \in A, b_\beta \in B\}$ a random subset is called *a unary-positional binary relation* on the sets A and B.

To demonstrate the difference between unary-positional and binary-positional binary relations we can have a look at the following example. On the set $Q = \{2, 3\}$ the unary-positional binary relations $q_\alpha > q_\alpha$ and $q_\beta > q_\beta$, where $q_\alpha, q_\beta \in Q$, are false (untrue), as it is shown in Table 1.

Table 1

The truth values of the unary-positional binary relation $q_\alpha > q_\alpha$

q_α		$q_\alpha > q_\alpha$
2_α	2_α	0
3_α	3_α	0

At the same time for the binary-positional binary relation $q_\alpha > q_\beta$, where $q_\alpha, q_\beta \in Q$, the truth values are presented in Table 2.

Table 2

The truth values of the binary-positional binary relation $q_\alpha > q_\beta$

q_α	q_β	$q_\alpha > q_\beta$
2_α	2_β	0
2_α	3_β	0
3_α	2_β	1
3_α	3_β	0

It is known [14] that a Cartesian product $A \times B \times \ldots \times W$ of sets A, B, ..., W is called the set $\{(a, b, \ldots, w) \mid a \in A, b \in B, \ldots, w \in W\}$, formed by the ordered sequences $(a, b, \ldots, w)$ of elements *a, b, ..., w.*

The difference between $q_\alpha > q_\alpha$ and $q_\alpha > q_\beta$ is that the values of the same variable in different positions can be the same and different. In general, all possible combinations of the variable values for all positions should be considered.

Definition 5. The unary-positional binary relation F, defined on the sets A and B, is called *a unary-positional functional one*, if for a random unary-positional ordered pair $(a_\alpha, b_\alpha) \in A \times B$ $((a_\beta, b_\beta) \in A \times B)$ there is no more than one element v from V that $(a_\alpha, b_\alpha) \in F$ $((a_\beta, b_\beta) \in F)$.

If there is such an element v from V for a unary-positional ordered pair (a_α, b_α) or (a_β, b_β), then it is denoted with the help of $F(a_\alpha, b_\alpha)$ or $F(a_\beta, b_\beta)$ and it is recorded as $v = F(a_\alpha, b_\alpha)$ or $v = F(a_\beta, b_\beta)$ respectively.

Definition 6. The binary-positional binary relation F, defined on the sets A and B, is called *a binary-positional functional one*, if for a random binary-positional ordered pair $(a_\alpha, b_\beta) \in A \times B$ there is no more than one element v from V that $(a_\alpha, b_\beta) \in F$.

If there is such an element v from V for a binary-positional ordered pair (a_α, b_β), then it is denoted with the help of $F(a_\alpha, b_\beta)$ and it is recorded as $v = F(a_\alpha, b_\beta)$.

Definition 7. A unary- or binary positional binary relation F is called *absolutely defined*, if $Dom(F) = A \times B$, and *partially defined* or just *partial*, if $Dom(F) \subset A \times B$.

The unary- or binary positional binary relation F, defined on the sets A and B, is called a *unary-* or *binary-positional binary representation,* or *a unary-* or *binary-positional function* from $A \times B$ into V $(F : A \times B \to V)$, if F is unary- or binary-positional functional and absolutely defined.

The unary- or binary positional binary relation F is called a *unary-* or *binary-positional partial representation or a unary-* or *binary-positional partial function* if F is unary- or binary-positional functional and partial.

Definition 8. If $F : A^2 \to V$, then F is a unary- or binary-positional binary function from A into V, and if $V = \{0, 1\}$ at the same time, then F is a unary- or binary-positional binary predicate on the set A, and the elements 0 and 1 are *false* and *true.*

If F is a unary- or binary-positional binary function from A^2 into A, then F is a unary- or binary-positional binary operation on A.

3. Definitions of Unary- and Binary-Positional Logical Operations

Definition 9. The unary-positional binary operation $(x_\alpha, y_\alpha)\underline{\&}$, which has the truth value of 1 for random positional variables $x_\alpha, y_\alpha, \in \{0, 1\}$ of the position α, if and only if $x_\alpha = 1_\alpha$ and $y_\alpha = 1_\alpha$, is called *α-conjunction.*

From the definition of α-conjunction, it follows that:

$$(x_\alpha, y_\alpha)\underline{\&} = \begin{cases} 1, \text{ if } x_\alpha = 1_\alpha \text{ and } y_\alpha = 1_\alpha; \\ 0 - \text{in all other cases,} \end{cases}$$

where x_α and y_α are positional variables of the elementary position α.

In a simplified form we record $(x_\alpha, y_\alpha)\underline{\&}$ as $(x, y)\underline{\&}$.

Definition 10. The unary-positional binary operation $\underline{\&}(x_\beta, y_\beta)$, which has the truth value of 1 for random positional variables $x_\beta, y_\beta \in \{0, 1\}$ of the position β if and only if $x_\beta = 1_\beta$ and $y_\beta = 1_\beta$, is called *β- conjunction.*

From the definition of β-conjunction, we get:

$$\underline{\&}(x_\beta, y_\beta) = \begin{cases} 1, \text{ if } x_\beta = 1_\beta \text{ and } y_\beta = 1_\beta; \\ 0 - \text{in all other cases,} \end{cases}$$

where x_β and y_β are positional variables of the elementary position β.

In a simplified form we record $\underline{\&}(x_\beta, y_\beta)$ as $\underline{\&}(x, y)$.

Definition 11. The binary-positional binary operation $<x_\alpha, y_\beta>$, which has the truth value of 1 for random positional variables $x_\alpha, y_\beta \in \{0, 1\}$ of the positions α and β if and only if $x_\alpha = 1_\alpha$, and $y_\beta = 1_\beta$, is called *αβ-conjunction.*

From the definition of $\alpha\beta$-conjunction, we have:

$$<x_\alpha, y_\beta> = \begin{cases} 1, \text{ if } x_\alpha = 1_\alpha \text{ and } y_\beta = 1_\beta; \\ 0 - \text{in all other cases.} \end{cases}$$

In a simplified form we record $<x_\alpha, y_\beta>$ as $<x, y>$.

Definition 12. The unary-positional binary operation $(x_\alpha, y_\alpha)\underline{\vee}$, which has the truth value of 1 for random positional variables $x_\alpha, y_\alpha \in \{0, 1\}$ of the position α if and only if $x_\alpha = 1_\alpha$ and $y_\alpha = 1_\alpha$ or if at least one positional variable has the truth value 1_α, is called *α-disjunction.*

Based on the definition of α-disjunction, we get:

$$(x_\alpha, y_\alpha)\underline{\vee} = \begin{cases} 1, \text{ if } x_\alpha = 1_\alpha \text{ and } y_\alpha = 1_\alpha, \text{ or } x_\alpha = 1_\alpha \text{ or } y_\alpha = 1_\alpha; \\ 0 - \text{in all other cases.} \end{cases}$$

The simplified record of $(x_\alpha, y_\alpha)\underline{\vee}$ is $(x, y)\underline{\vee}$.

Definition 13. The unary-positional binary operation $\underline{\vee}(x_\beta, y_\beta)$, which has the truth value of 1 for random positional variables $x_\beta, y_\beta \in \{0, 1\}$ of the position β if and only if $x_\beta = 1_\beta$ and $y_\beta = 1_\beta$ or if at least one positional variable has the truth value 1_β, is called *β-disjunction*.

Based on the definition of β-disjunction, we get:

$$\underline{\vee}(x_\beta, y_\beta) = \begin{cases} 1, \text{ if } x_\beta = 1_\beta \text{ and } y_\beta = 1_\beta, \text{ or } x_\beta = 1_\beta \text{ or } y_\beta = 1_\beta; \\ 0 - \text{in all other cases.} \end{cases}$$

$\underline{\vee}(x, y)$ is a simplified form of the record of β-disjunction $\underline{\vee}(x_\beta, y_\beta)$.

Definition 14. The binary-positional binary operation $|x_\alpha, y_\beta|$, which has the truth value of 1 for random positional variables $x_\alpha, y_\beta \in \{0, 1\}$ of the positions α and β if and only if $x_\alpha = 1_\alpha$ and $y_\beta = 1_\beta$ or $x_\alpha = 1_\alpha$ and $y_\beta = 1_\beta$, is called *αβ-disjunction*.

From the definition of $\alpha\beta$- disjunction, we get:

$$|x_\alpha, y_\beta| = \begin{cases} 1, \text{ if } x_\alpha = 1_\alpha \text{ and } y_\beta = 1_\beta, \text{ or } x_\alpha = 1_\alpha \text{ or } y_\beta = 1_\beta; \\ 0 - \text{in all other cases.} \end{cases}$$

$|x, y|$ – is a simplified form of the record of $\alpha\beta$-disjunction $|x_\alpha, y_\beta|$.

Definition 15. The unary-positional unary operation $\neg x_\varphi$, where φ φ is α or β position of the positional variable x_φ, has the truth value for any $x_\varphi \in \{0, 1\}$ when x_φ is the logical value 0_φ, then $\neg x_\varphi$ is the logical value 1, and when x_φ is the logical value 1_φ, then $\neg x_\varphi$ is the logical value 0_φ, is called *positional inverting*.

Definition 16. The formulas are:

i) logical constants and variables,

ii) positional inverting, α-, β-, $\alpha\beta$-conjunction and disjunction,

iii) if A and B are formulas, then $A = B$ is a formula,

iiii) any expression F is a formula, if it can be shown using the items *(i) – (iii)* with a finite number of times.

Theorem 1. The operation of α-conjunction is commutative.

Proof. The commutativity of the operation of α-conjunction means that there is an equality:

$$(x, y)\underline{\&} = (y, x)\underline{\&}. \tag{1}$$

The truth values of α-conjunctions $(x, y)\underline{\&}$ and $(y, x)\underline{\&}$ are presented in Table 3.

Table 3

The truth values of α-conjunction $(x, y)\underline{\&}$ and $(y, x)\underline{\&}$

x_α	y_α	$(x, y)\underline{\&}$	$(y, x)\underline{\&}$
0_α	0_α	0	0
0_α	1_α	0	0
1_α	0_α	0	0
1_α	1_α	1	1

The truth values of $(x, y)\underline{\&}$ and $(y, x)\underline{\&}$ in Table 3 are identical. It means that $(x, y)\underline{\&} = (y, x)\underline{\&}$.

The theorem has been proved.

Similarly, we prove the equalities:

$$\underline{\&}(x, y) = \underline{\&}(y, x), \tag{2}$$

$$(x, y)\vee = (y, x)\vee, \tag{3}$$

$$\vee(x, y) = \vee(y, x), \tag{4}$$

i.e. the operations of α-, β-conjunction and disjunction are commutative.

The truth of formulas:

$$(x, 1)\underline{\&} = \begin{cases} 1, \text{ if } x = 1; \\ 0 - \text{in all other cases;} \end{cases} \tag{5}$$

$$\underline{\&}(x, 1) = \begin{cases} 1, \text{ if } x = 1; \\ 0 - \text{in all other cases;} \end{cases} \tag{6}$$

$$<x, 1> = \begin{cases} 1, \text{ if } x = 1; \\ 0 - \text{in all other cases;} \end{cases} \tag{7}$$

$$<1, y> = \begin{cases} 1, \text{ if } y = 1; \\ 0 - \text{in all other cases;} \end{cases} \tag{8}$$

$$(x, 0)\underline{\vee} = \begin{cases} 0, \text{ if } x = 0; \\ 1 - \text{in all other cases;} \end{cases} \tag{9}$$

$$\underline{\vee}(x, 0) = \begin{cases} 0, \text{ if } x = 0; \\ 1 - \text{in all other cases;} \end{cases} \tag{10}$$

$$|x, 0| = \begin{cases} 0, \text{ if } x = 0; \\ 1 - \text{in all other cases;} \end{cases} \tag{11}$$

$$|0, y| = \begin{cases} 0, \text{ if } y = 0; \\ 1 - \text{in all other cases;} \end{cases} \tag{12}$$

follows directly from their definitions (see Definitions 9-14).

Definition 17. The sequence of elementary positions of positional logical constants and variables formed by operations is called *a position* of positional constants and variables.

For example, the formula $<x, (y, z)\underline{\&}>$ contains the operations of $\alpha\beta$-conjunction and α-conjunction. In α-conjunction $(y, z)\underline{\&}$ the variables y and z have the elementary position α and α, i.e. we have y_α and z_α. And α-conjunction itself has the elementary position β in the operation of $\alpha\beta$-conjunction and the variable x has the elementary position α. Taking into account the elementary position β of the operation of α-conjunction for positional variables y_α and z_α takes us to the positional variables $y_{\alpha\beta}$ and $z_{\alpha\beta}$. Thus, in the formula $<x, (y, z)\underline{\&}>$ the positional variables x, y and z have the positions α, $\alpha\beta$ і $\alpha\beta$.

In the operations, as a rule, the positions, which contain positional variables, will be omitted to simplify the formula recording.

Theorem 2. There is the distributivity:

$$<x, (y, z)\underline{\vee}> = (<x, y>, <x, z>)\underline{\vee}. \tag{13}$$

Proof. In the operation of $\alpha\beta$-conjunction $<x, y>$ the positional variables x and y have the elementary positions α and β, i.e. we have $<x_\alpha, y_\beta>$. Similarly, in $\alpha\beta$-

conjunction $<x, z>$ α and β are the elementary positions of s and y, respectively. Thus, we get the formula $<x_\alpha, z_\beta>$.

In the operation of α-disjunction $(<x, y>, <x, z>)\underline{\vee}$, both $\alpha\beta$-conjunction $<x, y>$ and $\alpha\beta$-conjunction $<x, z>$ have the elementary position α each. The explicit recording of these positions gives the formula $(<x, y>_\alpha, <x, z>_\alpha)\underline{\vee}$. Taking into account the positions of the variables in $\alpha\beta$-conjunctions $<x, y>$ and $<x, z>$, we get $(<x_\alpha, y_\beta>_\alpha, <x_\alpha, z_\beta>_\alpha)\underline{\vee}$. Adding the elementary position α of the operations $<x_\alpha, y_\beta>_\alpha$ and $<x_\alpha, z_\beta>_\alpha$ to the elementary positions of the positional variables x_α, y_β and z_β we get the positional variables $x_{\alpha\alpha}$, $y_{\beta\alpha}$ and $z_{\beta\alpha}$ with the positions $\alpha\alpha$, $\beta\alpha$ and $\beta\alpha$. Thus, the formula $(<x, y>, <x, z>)\underline{\vee}$ with explicitly recorded positions of the positional variables looks the following way $(<x_{\alpha\alpha}, y_{\beta\alpha}>, <x_{\alpha\alpha}, z_{\beta\alpha}>)\underline{\vee}$.

Similarly, we record explicitly the positions of the positional variables of the formula $<x, (y, z)\underline{\vee}>$. In the formula $(y, z)\underline{\vee}$ the positional variables y and z have the elementary positions α and α. So $(y, z)\underline{\vee}$ is the formula $(y_\alpha, z_\alpha)\underline{\vee}$. It has the elementary position β in the operation $<x_\alpha, (y_\alpha, z_\alpha)\underline{\vee}>$. Having recorded explicitly the elementary positions of the positional variable x and $(y_\alpha, z_\alpha)\underline{\vee}$ of the operation of $\alpha\beta$-conjunction $<x, (y_\alpha, z_\alpha)\underline{\vee}>$, we get the formula $<x_\alpha, (y_\alpha, z_\alpha)\underline{\vee}_\beta>$. Adding the position β to the elementary positions of the positional variables y_α and z_α, we have the formula $<x_\alpha, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$.

Based on the formula (9), we have $x = (x, 0)\underline{\vee}$. Substituting into the formula $<x_\alpha, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$ instead of x on the position α of the formula $(x, 0)\underline{\vee}$, we get $<(x, 0)\underline{\vee}_\alpha, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$. The variable x and the logical constant 0 in α-disjunction $(x, 0)\underline{\vee}_\alpha$ have the position α, so we have $(x_\alpha, 0_\alpha)\underline{\vee}_\alpha$. Taking the position α under the sign of the operation of α-disjunction, we get $(x_{\alpha\alpha}, 0_{\alpha\alpha})\underline{\vee}$. Substituting the received expression into the formula $<(x, 0)\underline{\vee}_\alpha, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$, we deduce the formula $<(x_{\alpha\alpha}, 0_{\alpha\alpha})\underline{\vee}, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$.

In $<(x_{\alpha\alpha}, 0_{\alpha\alpha})\underline{\vee}, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$ both operations of α-disjunction are *embedded* into the operation of $\alpha\beta$-conjunction. The variables x, y, z and the constant *0* in the embedded operations of α-disjunction have the elementary position α, i.e. we have x_α, y_α, z_α and 0_α. In the operation of $\alpha\beta$-conjunction the positional variable x_α and the positional constant 0_α have the elementary position α, and the positional variables y_α and z_α already have the elementary position β. If, when transforming the formulas, the embedding of the formulas needs to be changed, then the elementary positions of the variables and the constants of the operations must be kept as they were before the transformations. For example, the transformation of the formula $<(x_{\alpha\alpha}, 0_{\alpha\alpha})\underline{\vee}, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$ into the formula $(<x_{\alpha\alpha}, y_{\beta\alpha}>, <x_{\alpha\alpha}, z_{\beta\alpha}>)\underline{\vee}$ leads to the fact, that the operations of $\alpha\beta$-conjunction are already embedded into the operation of α-disjunction. Thus, in the formula received after the transformations $(<x_{\alpha\alpha}, y_{\beta\alpha}>, <x_{\alpha\alpha}, z_{\beta\alpha}>)\underline{\vee}$ the elementary positions of variables and constants of the operations are kept the same as they were in the formula $<(x_{\alpha\alpha}, 0_{\alpha\alpha})\underline{\vee}, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$ before the transformations. In addition, we can assume that the position of $\alpha\beta$ variables y and z in the formula $<(x_{\alpha\alpha}, 0_{\alpha\alpha})\underline{\vee}, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$ is the same as the position $\beta\alpha$ in the formula $(<x_{\alpha\alpha}, y_{\beta\alpha}>, <x_{\alpha\alpha}, z_{\beta\alpha}>)\underline{\vee}$.

Table 4 presents the truth values of the formulas $<(x_{\alpha\alpha}, 0_{\alpha\alpha})\underline{\vee}, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$ and $(<x_{\alpha\alpha}, y_{\beta\alpha}>, <x_{\alpha\alpha}, z_{\beta\alpha}>)\underline{\vee}$, taking into account the same truth values of the positional variables y and z on the positions $\alpha\beta$ and $\beta\alpha$.

Table 4

The truth values of the formulas $<(x_{\alpha\alpha}, 0_{\alpha\alpha})\underline{\vee}, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$ *and* $(<x_{\alpha\alpha}, y_{\beta\alpha}>, <x_{\alpha\alpha}, z_{\beta\alpha}>)\underline{\vee}$

$x_{\alpha\alpha}$	$y_{\beta\alpha}$	$z_{\beta\alpha}$	$y_{\alpha\beta}$	$z_{\alpha\beta}$	$<(x_{\alpha\alpha}, 0_{\alpha\alpha})\underline{\vee}, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$	$(<x_{\alpha\alpha}, y_{\beta\alpha}>, <x_{\alpha\alpha}, z_{\beta\alpha}>)\underline{\vee}$
$0_{\alpha\alpha}$	$0_{\beta\alpha}$	$0_{\beta\alpha}$	$0_{\alpha\beta}$	$0_{\alpha\beta}$	0	0
$0_{\alpha\alpha}$	$0_{\beta\alpha}$	$1_{\beta\alpha}$	$0_{\alpha\beta}$	$1_{\alpha\beta}$	0	0
$0_{\alpha\alpha}$	$1_{\beta\alpha}$	$0_{\beta\alpha}$	$1_{\alpha\beta}$	$0_{\alpha\beta}$	0	0
$0_{\alpha\alpha}$	$1_{\beta\alpha}$	$1_{\beta\alpha}$	$1_{\alpha\beta}$	$1_{\alpha\beta}$	0	0
$1_{\alpha\alpha}$	$0_{\beta\alpha}$	$0_{\beta\alpha}$	$0_{\alpha\beta}$	$0_{\alpha\beta}$	0	0
$1_{\alpha\alpha}$	$0_{\beta\alpha}$	$1_{\beta\alpha}$	$0_{\alpha\beta}$	$1_{\alpha\beta}$	1	1
$1_{\alpha\alpha}$	$1_{\beta\alpha}$	$0_{\beta\alpha}$	$1_{\alpha\beta}$	$0_{\alpha\beta}$	1	1
$1_{\alpha\alpha}$	$1_{\beta\alpha}$	$1_{\beta\alpha}$	$1_{\alpha\beta}$	$1_{\alpha\beta}$	1	1

Table 4 shows, that the truth values of the formulas $<(x_{\alpha\alpha}, 0_{\alpha\alpha})\underline{\vee}, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}>$and $(<x_{\alpha\alpha}, y_{\beta\alpha}>, <x_{\alpha\alpha}, z_{\beta\alpha}>)\underline{\vee}$ are identical. Taking into account that $(x_{\alpha\alpha}, 0_{\alpha\alpha})\underline{\vee} = x_{\alpha}$ (Formula 9), it proves the truth of the formula:

$$<x_{\alpha}, (y_{\alpha\beta}, z_{\alpha\beta})\underline{\vee}> = (<x_{\alpha\alpha}, y_{\beta\alpha}>, <x_{\alpha\alpha}, z_{\beta\alpha}>)\underline{\vee}.$$

Omitting the positions in the received formula, we get a simplified form:

$$<x, (y, z)\underline{\vee}> = (<x, y>, <x, z>)\underline{\vee}.$$

The theorem has been proved.

Similarly, we can prove that:

$$<x, \underline{\vee}(x, y)> = \underline{\vee}(<x, y>, <x, z>). \quad (14)$$

Theorem 3. There is the equality:

$$\underline{\neg}(x, y)\underline{\&} = (\underline{\neg}x, \underline{\neg}y)\underline{\vee}. \quad (15)$$

Proof. Taking into account the definition of the operation of positional inverting (see Definition 13), the truth values of the inverted operation of α-conjunction $\underline{\neg}(x, y)\underline{\&}$ will be nothing else but the inverted truth values of α-conjunction $(x, y)\underline{\&}$, i.e. the truth values will be identical to the ones in Table 5.

Table 5

The inverted truth values of α-conjunction $(x, y)\underline{\&}$

x_{α}	y_{α}	$(x, y)\underline{\&}$	$\underline{\neg}\,(x, y)\underline{\&}$
0_{α}	0_{α}	0	1
0_{α}	1_{α}	0	1
1_{α}	0_{α}	0	1
1_{α}	1_{α}	1	0

Table 6 presents the truth values of the formula $(\underline{\neg}x, \underline{\neg}y)\underline{\vee}$.

Table 6

The truth values of the formula $(\sqsupset x, \sqsupset y)\underline{\vee}$

x_α	y_α	$(\sqsupset x, \sqsupset y)\underline{\vee}$
0_α	0_α	1
0_α	1_α	1
1_α	0_α	1
1_α	1_α	0

Tables 5 and 6 have the identical truth values for α-disjunction $(\sqsupset x, \sqsupset y)\underline{\vee}$ and inverted α-conjunction $\sqsupset(x, y)\underline{\&}$. Based on this fact, we have established that $\sqsupset(x, y)\underline{\&} = (\sqsupset x, \sqsupset y)\underline{\vee}$.

It is obvious that:

$$\sqsupset(x, y)\underline{\&} = (\sqsupset x_\alpha, \sqsupset y_\alpha)\underline{\vee} = \begin{cases} 0, \text{ if } x = x_\alpha = 1_\alpha \text{ and } y = y_\alpha = 1_\alpha; \\ 1 - \text{in all other cases,} \end{cases}$$

The theorem has been proved.

Similarly, we can prove the equalities:

$$\sqsupset\&(x_\beta, y_\beta) = \underline{\vee}(\sqsupset x_\beta, \neg y_\beta), \tag{16}$$

$$\sqsupset\langle x_\alpha, y_\beta\rangle = |\sqsupset x_\alpha; \sqsupset y_\beta|, \tag{17}$$

$$\sqsupset(x_\alpha, y_\alpha)\underline{\vee} = (\sqsupset x_\alpha, \sqsupset y_\alpha)\&, \tag{18}$$

$$\sqsupset\vee(x_\beta, y_\beta) = \&(\sqsupset x_\beta, \sqsupset y_\beta), \tag{19}$$

$$\sqsupset|x_\alpha, y_\beta| = \langle\sqsupset x_\alpha, \sqsupset y_\beta\rangle. \tag{20}$$

Theorem 4. The operation of α-conjunction is associative:

$$((x, y)\underline{\&}, z)\underline{\&} = (x, (y, z)\underline{\&})\underline{\&} \tag{21}$$

Proof. The formulas $((x, y)\underline{\&}, z)\underline{\&}$ and $(x, (y, z)\underline{\&})\underline{\&}$ with explicitly recorded positions of the positional variables x, y and z look as $((x_{\alpha\alpha}, y_{\alpha\alpha})\underline{\&}, z_\alpha)\underline{\&}$ and $(x_\alpha, (y_{\alpha\alpha}, z_{\alpha\alpha})\underline{\&})\underline{\&}$.

Taking into account that $(x, 1)\underline{\&} = x$ (formula (6)) and $(z, 1)\underline{\&} = z$, we can deduce the formulas $((x_{\alpha\alpha}, y_{\alpha\alpha})\underline{\&}, (z, 1)\underline{\&}_{\alpha\alpha})\underline{\&}$ and $((x, 1)\underline{\&}_{\alpha\alpha}, (y_{\alpha\alpha}, z_{\alpha\alpha})\underline{\&})\underline{\&}$. The explicit record of all positions gives the formulas $((x_{\alpha\alpha}, y_{\alpha\alpha})\underline{\&}, (z_{\alpha\alpha}, 1_{\alpha\alpha})\underline{\&})\underline{\&}$ and $((x_{\alpha\alpha}, 1_{\alpha\alpha})\underline{\&}, (y_{\alpha\alpha}, z_{\alpha\alpha})\underline{\&})\underline{\&}$. Both formulas are formed by the same positional variables, over which we perform only the operation of α-conjunction, so $((x_{\alpha\alpha}, y_{\alpha\alpha})\underline{\&}, (z_{\alpha\alpha}, 1_{\alpha\alpha})\underline{\&})\underline{\&} = ((x_{\alpha\alpha}, 1_{\alpha\alpha})\underline{\&}, (y_{\alpha\alpha}, z_{\alpha\alpha})\underline{\&})\underline{\&}$, and thus $((x, y)\underline{\&}, z)\underline{\&} = (x, (y, z)\underline{\&})\underline{\&}$.

The theorem has been proved.

Similarly, we prove the truth of the formula:

$$((x, y)\underline{\vee}, z)\underline{\vee} = (x, (y, z)\underline{\vee})\underline{\vee}. \tag{22}$$

4. Representation of classical logical operations of conjunction and disjunction with the operations of *α*-, *β*-, *αβ*-conjunction (-disjunction)

To establish the relations between classical operations of conjunction and disjunction [1] and the operations of *α*-, *β*-, *αβ*-conjunctions (-disjunctions), we will formulate and prove the theorems.

Theorem 5. The classical operation of conjunction $x \,\&\, y$ is equal to the classical disjunction of the operations of *α*-conjunction $(x, y)\underline{\&}$, *β*-conjunction $\underline{\&}(x, y)$, *αβ*-conjunction $<x, y>$, and *αβ*-conjunction $<y, x>$.

Proof. Table 7 shows the truth values of *α*-, *β*-, *αβ*-conjunctions represented by the formulas $(x, y)\underline{\&}$, $\underline{\&}(x, y)$, $<x, y>$ and $<y, x>$, respectively, and the classical conjunction $x \,\&\, y$.

Table 7

The truth values $(x, y)\underline{\&}$, $\underline{\&}(x, y)$, $<x, y>$, $<y, x>$ *and* $x \,\&\, y$

α		*β*		$(x, y)\underline{\&}$	$\underline{\&}(x, y)$	$<x, y>$	$<y, x>$	$x \,\&\, y$
x	*y*	*x*	*y*					
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	0	1	1	0	1	0	0	1
0	1	0	0	0	0	0	0	0
0	1	0	1	0	0	0	0	0
0	1	1	0	0	0	0	1	1
0	1	1	1	0	1	0	1	1
1	0	0	0	0	0	0	0	0
1	0	0	1	0	0	1	0	1
1	0	1	0	0	0	0	0	0
1	0	1	1	0	1	1	0	1
1	1	0	0	1	0	0	0	1
1	1	0	1	1	0	1	0	1
1	1	1	0	1	0	0	1	1
1	1	1	1	1	1	1	1	1

The classical operation of conjunction $x \,\&\, y$ takes the truth value 1 if and only if both variables x and y have the truth values 1, regardless of their positions, as shown in Table 7.

Looking at Table 7, it is easy to see that there is the equality:

$$x \,\&\, y = (x, y)\underline{\&} \vee \underline{\&}(x, y) \vee <x, y> \vee <y, x>.$$

The theorem has been proved.

Theorem 6. The classical operation of disjunction $x \vee y$ is equal to the classical disjunction of the operation of *α*-disjunction $(x, y)\underline{\vee}$, *β*-disjunction $\underline{\vee}(x, y)$, *αβ*-disjunction $|x, y|$ and *αβ*-disjunction $|y, x|$.

Proof. Table 8 presents the truth values of *α*-, *β*-, *αβ*-disjunctions, represented by the formulas $(x, y)\underline{\vee}$, $\underline{\vee}(x, y)$, $|x, y|$ and $|y, x|$, respectively, and the classical disjunction $x \vee y$.

Table 8

The truth values $(x, y)\underline{\vee}$, $\underline{\vee}(x, y)$, $|x, y|$, $|y, x|$ *and* $x \vee y$

α		β		$(x, y)\underline{\vee}$	$\underline{\vee}(x, y)$	$\|x, y\|$	$\|y, x\|$	$x \vee y$
x	y	x	y					
0	0	0	0	0	0	0	0	0
0	0	0	1	0	1	1	0	1
0	0	1	0	0	1	0	1	1
0	0	1	1	0	1	1	1	1
0	1	0	0	1	0	0	1	1
0	1	0	1	1	1	1	1	1
0	1	1	0	1	1	0	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	1	0	1
1	0	0	1	1	1	1	0	1
1	0	1	0	1	1	1	1	1
1	0	1	1	1	1	1	1	1
1	1	0	0	1	0	1	1	1
1	1	0	1	1	1	1	1	1
1	1	1	0	1	1	1	1	1
1	1	1	1	1	1	1	1	1

Based on Table 8 we have the equality:

$$x \vee y = (x, y)\underline{\vee} \vee \underline{\vee}(x, y) \vee |x, y| \vee |y, x|.$$

The theorem has been proved.

5. Ordering

The positional elements of the operations of α-, *β*-, *αβ*-conjunction (disjunction) and positional inverting are positional constants or variables. For example, 0_α, 1_β, x_α, y_β are positional elements of the operations. Then we denote the positional elements of the operations by capital letters of the Latin alphabet.

Definition 18. A random formula F, each position of positional elements of the operation of which differs from the other positions of positional elements is called *strictly ordered.*

Theorem 7. A random formula F, formed by *αβ*-conjunctions and (or) *αβ*-disjunctions, is strictly ordered.

Proof. There are 14 possible different basic options for constructing the formulas by *αβ*-conjunctions and (or) *αβ*-disjunctions, and all other options are combinations of these 14 options. The basic ones are:

1) $<B_\alpha, C_\beta>$ – *αβ*-conjunction of positional elements;
2) $<B_\alpha, <C_\alpha, D_\beta>_\beta>$ – *αβ*-conjunction of a positional element and *αβ*-conjunction;
3) $<<B_\alpha, C_\beta>_\alpha, D_\beta>$ – *αβ*-conjunction of *αβ*-conjunction and a positional element;
4) $<<B_\alpha, C_\beta>_\alpha, <D_\alpha, E_\beta>_\beta>$ – *αβ*-conjunction of *αβ*-conjunctions;
5) $|B_\alpha, C_\beta|$ – *αβ*-disjunction of positional elements;
6) $|B_\alpha, |C_\alpha, D_\beta|_\beta|$ – *αβ*-disjunction of a positional element and *αβ*-disjunction;
7) $||B_\alpha, C_\beta|_\alpha, D_\beta|$ – *αβ*-disjunction of *αβ*-disjunction and a positional element;

8) $||B_\alpha, C_\beta|_\alpha, |D_\alpha, E_\beta|_\beta|$ – *αβ*-disjunction of *αβ*-disjunctions;

9) $<B_\alpha, |C_\alpha, D_\beta|_\beta>$ – *αβ*-conjunction of a positional element and *αβ*-disjunction;

10) $<|B_\alpha, C_\beta|_\alpha, D_\beta>$ – *αβ*-conjunction of *αβ*-disjunction and a positional element;

11) $<|B_\alpha, C_\beta|_\alpha, |D_\alpha, E_\beta|_\beta>$ – *αβ*-conjunction of *αβ*-disjunctions;

12) $|B_\alpha, <C_\alpha, D_\beta>_\beta|$ – *αβ*-disjunction of a positional element and *αβ*-conjunction;

13) $|<B_\alpha, C_\beta>_\alpha, D_\beta|$ – *αβ*-disjunction of *αβ*-conjunction and a positional element;

14) $|<B_\alpha, C_\beta>_\alpha, <D_\alpha, E_\beta>_\beta|$ – *αβ*-disjunction of *αβ*-conjunctions, where B_α, C_α, C_β, D_α, D_β and E_β are positional elements with elementary positions. These positional elements are positional constants or variables.

Assigning the elementary positions of operations to the elementary positions of their positional elements, we get the formulas:

1) $<B_\alpha, C_\beta>$ – *αβ*-conjunction of positional elements;

2) $<B_\alpha, <C_{\alpha\beta}, D_{\beta\beta}>>$ – *αβ*-conjunction of a positional element and *αβ*-conjunction;

3) $<<B_{\alpha\alpha}, C_{\beta\alpha}>, D_\beta>$ – *αβ*-conjunction of *αβ*-conjunction and a positional element;

4) $<<B_{\alpha\alpha}, C_{\beta\alpha}>, <D_{\alpha\beta}, E_{\beta\beta}>>$ – *αβ*-conjunction of *αβ*-conjunctions;

5) $|B_\alpha, C_\beta|$ – *αβ*-disjunction of positional elements;

6) $|B_\alpha, |C_{\alpha\beta}, D_{\beta\beta}||$ – *αβ*-disjunction of a positional element and *αβ*-disjunction;

7) $||B_{\alpha\alpha}, C_{\beta\alpha}|, D_\beta|$ – *αβ*-disjunction of *αβ*-disjunction and a positional element;

8) $||B_{\alpha\alpha}, C_{\beta\alpha}|, |D_{\alpha\beta}, E_{\beta\beta}||$ – *αβ*-disjunction of *αβ*-disjunctions;

9) $<B_\alpha, |C_{\alpha\beta}, D_{\beta\beta}|>$ – *αβ*-conjunction of a positional element and *αβ*-disjunction;

10) $<|B_{\alpha\alpha}, C_{\beta\alpha}|, D_\beta>$ – *αβ*-conjunction of *αβ*-disjunction and a positional element;

11) $<|B_{\alpha\alpha}, C_{\beta\alpha}|, |D_{\alpha\beta}, E_{\beta\beta}|>$ – *αβ*-conjunction of *αβ*-disjunctions;

12) $|B_\alpha, <C_{\alpha\beta}, D_{\beta\beta}>|$ – *αβ*-disjunction of a positional element and *αβ*-conjunction;

13) $|<B_{\alpha\alpha}, C_{\beta\alpha}>, D_\beta|$ – *αβ*-disjunction of *αβ*-conjunction and a positional element;

14) $|<B_{\alpha\alpha}, C_{\beta\alpha}>, <D_{\alpha\beta}, E_{\beta\beta}>|$ – *αβ*-disjunction of *αβ*-conjunctions.

Each of the 14 basic formula options has positional elements, each position of which is different from all other positions of positional elements. For example, the basic formula of the second option is formed by the positional elements B_α, $C_{\alpha\beta}$, $D_{\beta\beta}$ with positions *α*, *αβ* and *ββ*, which differ from each other. Similarly, Formula 14 is formed by the positional elements $B_{\alpha\alpha}$, $C_{\beta\alpha}$, $D_{\alpha\beta}$ and $E_{\beta\beta}$, whose positions are also different. The basic formulas are strictly ordered. All other formulas constructed from them will also be strictly ordered.

The theorem has been proved.

The rule of ordering of positional elements by positions.

The *initial* positional element of a random strictly ordered formula *F* is:

- a positional element with the elementary position *α*;
- if there is no such a positional element, then the initial one is a positional element whose position is formed only by the elementary positions *α* and the number of these positions is the smallest;
- if there are several positional elements with the same number of elementary positions *α*, then any of them is initial.

Examples. In a strictly ordered formula:

$$<X, <Y, <Z, <R, <S, T>>>>>$$

the initial element is the positional element *X*, as it has the elementary position α. Whereas the formulas:

$$<<<A, B>, X>, <Y, <Z, <R, <S, T>>>>> \text{ and } <<S, <R, <Z, <Y, X>>>>, T>$$

have the initial positional elements *A* and *S* respectively.

The *next* positional element is:

- the one which forms a binary-positional binary operation directly with the previous one;
- or it has the elementary position α in the binary-positional binary operation, which directly enters the binary-positional binary operation with the previous positional elements;
- or it has the position which is formed only by the elementary positions α in the binary-positional binary operations that are part of the binary-positional binary operation with the previous positional element.

Example 1. For a strictly ordered formula $<X, <Y, <Z, <R, <S, T>>>>>$ the next elements after the initial one is the element *Y*, and then the next after *Y* is the positional element *Z*, and then the next are the positional elements *R, S* and *T,* respectively.

Example 2. In strictly ordered formulas $<<<A, B>, X>, <Y, <Z, <R, <S, T>>>>>$ and $<<S, <R, <Z, <Y, X>>>>, T>$ the next ones are the positional elements *B*, *X*, *Y*, *Z*, *R*, *S*, *T* and *R*, *Z*, *Y*, *X*, *T* respectively.

Example 3. Let us have a bit complicated formula:

$$<X, <|A, |<<B, R>, S>, C||, Z>>$$

The positional element *X* is the first. *A* is the second. Then – *B*, and after it – *R* and then – *S*. Now – *C* and finally – *Z*.

Example 4. Let us have a look at the ordering of positional elements of the formulas *Q*, *D* and *M:*

$$Q = <X, <|D, |<<B, R>, S>, C||, Z>>,$$

$$D = <K, <L, |M, N|>>,$$

$$M = |P, <I, |J, <G, |H, |V, <W, E>||>|>|$$

Let the first positional element belong to the formula *Q*. This is *X*. The second is the positional element *D*. Since *D* is a formula, then we have *K* instead of *D*. After that we have *L* and then – *M*. Instead of *M* we have sequentially ordered positional elements *P*, *I*, *J*, *G*, *H*, *V*, *W* and *E*. Then – *N*. Later *B*, *R* and *S*. After that – *C* and finally – *Z*.

Other rules for ordering positional elements of formulas are also possible. Below there is one more.

The rule of ordering of positional elements by selecting the initial positional element.

Its essence lies in setting a positional element for any formula as the initial one. Each subsequent positional element is determined on the basis of the formation of the binary-positional binary operation with a previous positional element.

Example 5. Let for the formula:

$$<A, <B, ||C, <<D, G>, H>|, E|>>,$$

the positional element G is initial. It forms $\alpha\beta$-conjunction with the positional element D. Thus, D is the next positional element after G. $\alpha\beta$-conjunction $<D, G>$ together with H forms new $\alpha\beta$-conjunction. So, H is the next positional element after D. Now $\alpha\beta$-conjunction $<<D, G>, H>$ together with C forms $\alpha\beta$-disjunction $|C, <<D, G>, H>|$. Thus, C is the next positional element after H. Similarly the positional element E and $\alpha\beta$-disjunction $|C, <<D, G>, H>|$ form new $\alpha\beta$-disjunction $||C, <<D, G>, H>|, E|$. Thus, E is the next positional elements of the ordering. After that the positional element B enters the ordering, which with $||C, <<D, G>, H>|, E|$ forms the operation of $\alpha\beta$-conjunction $<B, ||C, <<D, G>, H>|, E|>$. The last in the ordering is the positional element A, because it with $<B, ||C, <<D, G>, H>|, E|>$ forms the operation of $\alpha\beta$-conjunction.

6. Conclusions

By the completed analysis of both classical and non-classical mathematical logic as well as the algebraic methods of description of algorithms, it has been found out that their operations do not set positions and they do not operate positions. The consideration of positions is extremely important in both their theoretical and applied value for an adequate description and transformations of ordered processes.

New operations of α-, β- and $\alpha\beta$-conjunction (disjunction) have been defined which assign α-, β- and α-β positions to the formulas and transform the sequences of positions.

The commutativity and associativity of α- and β- conjunction (disjunction) have been proved.

The distributivity of α-, β- and $\alpha\beta$-conjunction has been proved.

The operation of positional inverting has been introduced, the links between the operations of α-, β- and $\alpha\beta$-conjunction (disjunction) and positional inverting have been established.

The correctness of the introduced operations of α-, β-, $\alpha\beta$-conjunctions (-disjunctions) has been proved on the basis of establishing a mutual unambiguous correspondence between the classical operations of conjunction (disjunction) and the introduced new operations of α-, β-, $\alpha\beta$-conjunctions (-disjunctions).

It has been proved that a random formula formed by $\alpha\beta$-conjunctions (disjunctions) is strictly ordered.

The possibility of performing the identical transformations over the ordered formulas has been shown.

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ORDERING BY BINARY-POSITIONAL LOGICAL OPERATIONS

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The article shows that the operations of mathematical logic and algebraic methods of description of algorithms based on mathematical logic do not take into account the positions. New operations of α-, β-, αβ-conjunction (disjunction) and positional inverting have been defined, which take into account the positions. The properties of these operations have been formulated and proved. The mutual unambiguousness has been established between the classical operations of conjunction (disjunction) and the operations of α-, β-, αβ-conjunctions (-disjunctions). The ordering of formulas by positions and the possibility of performing identical transformations of the ordering have been proved.

Keywords: *α-conjunction, β-conjunction, αβ-conjunction, α-disjunction, β-disjunction, αβ-disjunction, positional inverting.*

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main types of motion of the UAV body, namely the hull displacement and the hull vibration caused by the rotation of the screws and subsequent processing of these signals and filtering noise. After the UAV is identified in the airspace, measures are taken to neutralize it, using the method of intercepting control or creating noise at the control frequencies.

Results. The suggested system for detecting the presence of drones, based on the identification and assessment of the mutual wavelet spectrum, has been determined on the basis of the information received about the vibration of the drone hull and its vibrations, which are transmitted from the unmanned aerial vehicle.

Novelty. The use of a small-field region for the conversion of input signals provides high efficiency of filtering and identification of signals, significantly improves the resolution and transmission coefficient in the spatial region of the small-field conversion.

Practical Significance. This development significantly reduces the price of the drone identification system. Also, the system has the ability to effectively identify a large number of drones at the same time.

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ORDERING BY BINARY-POSITIONAL LOGICAL OPERATIONS

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Research Methodology. The methodology of mathematical logic and the theory of sets have been used in the research.

Results. The most significant theoretical result is the creation of a new methodology, which is based on the introduction of positions in the binary Cartesian product of sets. The operations of mathematical logic and algebraic methods of description of algorithms based on mathematical logic do not take into account the positions. New operations of *α*-, *β*-, *αβ*-conjunction (disjunction) and positional inverting have been defined, which take into account the positions. The properties of these operations have been formulated and proved. The mutual unambiguousness has been established between the classical operations of conjunction (disjunction) and the operations of α-, β-, αβ-conjunctions (-disjunctions). The ordering of formulas by

positions and the possibility of performing identical transformations of the ordering have been proved.

Novelty. New, in addition to the methodology, is the created positional logic, which contains new operations α-, β-, αβ-conjuncture (disjunction) and positional inverting, which take into account the positions of their constituents.

Practical Significance. Practical significance of the work lies in the analytical description of ordering, in particular algorithms.

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UNIFORM APPROXIMATION BY RATIONAL EXPRESSION

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Research Methodology. The research of the method of calculating the parameters of uniform approximation by rational expression is based on the application of methods of mathematical analysis and the theory of approximation of functions.

Results. The method of calculating the parameters of uniform approximation by a rational expression has been developed.

Novelty. The algorithm of uniform approximation by rational expression has been described. The idea of the method is based on the construction of the power-average approximation as boundary approximation in norm under . It consists in constructing a boundary power-average approximation. The method of least squares with two variable weight functions is used to construct power-average approximations. First weight function provides the construction of a power-average approximation, and the second – specification the parameters linearized of rational expression. The method of successive refinement of weight functions has been suggested.

Practical Significance. An effective method for determining the parameters of uniform approximation by a rational expression has been suggested. Approximation of rational expression is used in constructing models of functional converters, modeling of control systems.