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FOUNDATIONS OF LOGIC OF ORDERINGS

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The article shows that logical constants and variables are not ordered by the operations of classical mathematical logic. α-, β-, αβ-, βα-conjunctions (disjunctions) and positional inverting have been identified which order logical constants, variables and predicates. The properties of α-, β-, αβ-, βα-conjunctions (disjunctions) and positional inverting have been presented.

Keywords: *α-conjunction, β-conjunction, αβ-conjunction, βα-conjunction, α-disjunction, β- disjunction, αβ- disjunction, βα- disjunction, positional inverting.*

1. Introduction

N-ary relations [1] and n-ary predicates [1] identified on a random set are in a one-to-one correspondence [1]. In this case, it is necessary to have tools for solving the problem of direct ordering of predicates that operate with logical values and take into account the clear positions of the predicates.

A position of something is its location or placing. The term is used in mathematics, computer sciences, informatics and many other areas. For example, mathematical induction [1, 2] uses the initial value of a variable, and programming and information technology have an initial instruction [3], which, like any other programming instruction, has a unique number or name. The “*initial*” value, as well as the “*numbers*” of the instructions, are actually their positions. In algorithms, the positions of operators have key values.

The operations of classical mathematical logic [1, 2], in particular conjunction (&) and disjunction (∨), operate with logical values. However, the positions of logical constants, variables and n-ary predicates are not set and are not taken into account. This also applies to non-classical mathematical logic, in particular the propositional modal logic, the propositional dynamic logic, the linear propositional temporal logic, the multivalued logic, the fuzzy logic, the intuitionistic logic, and so on.

In the general case, the orderings are non-commutative and non-associative. But the operations of conjunction (&) and disjunction (∨) are commutative and associative [1, 2], which also does not allow their application for the description of logical orderings.

Particularly important is the consideration of positions in mathematical logic, as the basis of modern mathematics. To obtain an adequate description and performing of identical transformations of algorithms, and not only algorithms, it is necessary to take into account the positions of operators, which form algorithms and applications

This paper is dedicated to solving the problem of ordering of logical elements by introducing and taking into account the clear positions of logical constants and variables.

2. *α*-, *β*-, *αβ*-, *βα*-conjunctions and -disjunctions

The set $B = \{0, 1, *\}$ is called *the set of logical values.* The element $0 \in B$ is logical with the value *false* (*logical constant* 0), $1 \in B$ is logical with the value *true* (*logical constant* 1), and $* \in B$ the value is undefined. $x, y, \dots$ are *logical variables*, $x, y, \dots \in B$.

The set $\Pi = \{\alpha, \beta\}$ is called *the set of elementary positions,* and $\alpha \in \Pi$ and $\beta \in \Pi$ *are elementary positions*. $\phi, \phi, \gamma, \delta, \dots$ are *variables of elementary positions*, $\phi, \psi, \gamma, \delta, \dots \in \Pi$.

A Cartesian product [2] $B \times \Pi$ of the sets B and Π form the set $P = \{(x, \phi) \mid x \in B, \phi \in \Pi\}$, which is called *the set of pairs*.

$P \times P$ – a Cartesian square of the set P forms the set $Q = \{[(x, \phi), (y, \psi)] \mid (x, \phi) \in P, (y, \psi) \in P; x, y \in B; \phi, \psi \in \Pi\}$, which is called *the set of ordered pairs.*

$P^2 \rightarrow B$ is the representation [2] of the Cartesian product $P \times P$ in B.

Definition 1. The representation $\&[(x, \phi), (y, \psi)]: P \times P \rightarrow B$ is the one that:

$$\&[(x, \varphi), (y, \psi)] = \begin{cases} 1, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \psi = \alpha, \text{ if } x, y \in B \text{ and } x = y = 1; \\ 0, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \psi = \alpha, \text{ if } x, y \in B, \\ \quad \text{and } x = y = 0, \text{ or } x = 0 \text{ and } y = 1, \text{ or } x = 1 \text{ and } y = 0; \\ *, \text{ in all other cases,} \end{cases}$$

and it is called *α-conjunction.*

The truth values of *α*-conjunction for logical variables *x*, *y* and the variables of the elementary positions ϕ, ψ are presented in Table 1.

Table 1

The truth values of α-conjunction &[(x, ϕ), (y, ψ)]

x / *y*			ϕ					
	α			*β*				
	0	1	*	0	1	*		
ψ	*α*	0	0	0	*	*	*	*
		1	0	1	*	*	*	*
		*	*	*	*	*	*	*
	β	0	*	*	*	*	*	*
		1	*	*	*	*	*	*
		*	*	*	*	*	*	*

In brief, α-conjunction $\&[(x, \phi), (y, \psi)]$ is recorded as $\&(x, y)$.

From the table above we see that for completely defined conditions (for $\phi = \psi = \alpha$ and the truth values 0 and 1 of the logical variables x and y) α-conjunction $\&(x, y)$ has the same logical truth values as a classical conjunction [1].

However, α-conjunction $\&(x, y)$ does not follow the law of contradiction $\&(x, \neg x) = 0$ and, besides, it sets the position α to the logical variable x as well as it sets the position α to the logical variable y.

Definition 2. The representation $[(x, \phi), (y, \psi)]\& : P \times P \to B$ is the one that:

$$[(x, \varphi), (y, \psi)]\& = \begin{cases} 1, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \psi = \beta, \text{ if } x, y \in B \text{ and } x = y = 1; \\ 0, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \psi = \beta, \text{ if } x, y \in B, \\ \quad \text{and } x = y = 0, \text{ or } x = 0 \text{ and } y = 1, \text{ or ж } x = 1 \text{ and } y = 0; \\ *, \text{ in all other cases,} \end{cases}$$

and it is called *β-conjunction*.

The truth values of β-conjunction for logical variables x, y and the variables of the elementary positions ϕ, ψ are presented in Table 2.

Table 2

The truth values of β-conjunction[(x, ϕ), (y, ψ)]&

x / y			ϕ					
	α			β				
	0	1	*	0	1	*		
ψ	α	0	*	*	*	*	*	*
		1	*	*	*	*	*	*
		*	*	*	*	*	*	*
	β	0	*	*	*	0	0	*
		1	*	*	*	0	1	*
		*	*	*	*	*	*	*

β-conjunction $[(x, \phi), (y, \psi)]\&$ is recorded as $(x, y)\&$ in brief.

From the table above we see that for completely defined conditions (for $\phi = \psi = \beta$ and the truth values 0 and 1 of the logical variables x and y) β-conjunction $(x, y)\&$ has the same logical truth values as a classical conjunction [1].

However, β- conjunction $(x, y)\&$ does not follow the law of contradiction $(x, \neg x)\& == 0$ and, besides, it sets the elementary position β to the logical variable x as well as it sets the elementary position β to the logical variable y.

Definition 3. The representation $<(x, \phi), (y, \psi)> : P \times P \to B$ is the one that:

$$<(x, \varphi), (y, \psi)> = \begin{cases} 1, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \alpha, \psi = \beta, \text{ if } x, y \in B \text{ and } x = y = 1; \\ 0, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \alpha, \psi = \beta, \text{ if } x, y \in B \\ \qquad \text{and } x = y = 0, \text{ or } x = 0 \text{ and } y = 1, \text{ or } x = 1 \text{ and } y = 0; \\ *, \text{ in all other cases,} \end{cases}$$

and it is called *αβ-conjunction*.

The truth values of *αβ*- conjunction for logical variables x, y and the variables of the elementary positions ϕ, ψ are presented in Table 3.

Table 3

The truth values of αβ- conjunction <(x, ϕ), (y, ψ)>

x / *y*			ϕ					
	α			β				
	0	1	*	0	1	*		
ψ	*α*	0	*	*	*	*	*	*
		1	*	*	*	*	*	*
		*	*	*	*	*	*	*
	β	0	0	0	*	*	*	*
		1	0	1	*	*	*	*
		*	*	*	*	*	*	*

In brief, *αβ*- conjunction $<(x, \phi), (y, \psi)>$ is recorded as $<x, y>$.

From the presented above definition it follows that for completely defined conditions (for $\phi = \alpha$ and $\psi = \beta$ and the truth values 0 and 1 of the logical variables x and y) *αβ*-conjunction $<x, y>$ has the same logical truth values as a classical conjunction [1].

However, *αβ*- conjunction $<x, y>$ does not follow the law of contradiction $<x, \neg x> == 0$ and, besides, it sets the elementary position α to the logical variable x as well as it sets the elementary position β to the logical variable y.

Definition 4. The representation $\wedge[(x, \phi), (y, \psi)] : P \times P \rightarrow B$ is the one that:

$$\wedge[(x, \varphi), (y, \psi)] = \begin{cases} 1, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \beta, \psi = \alpha, \text{ if } x, y \in B \text{ and } x = y = 1; \\ 0, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \beta, \psi = \alpha, \text{ if } x, y \in B \\ \qquad \text{and } x = y = 0, \text{ or } x = 0 \text{ and } y = 1, \text{ or } x = 1 \text{ and } y = 0; \\ *, \text{ in all other cases,} \end{cases}$$

and it is called *βα-conjunction*. The truth values of *βα*-conjunction for logical variables x, y and the variables of the elementary positions ϕ, ψ are presented in Table 4.

Table 4

The truth values of βα-conjunction

x / y			ϕ					
	α			β				
	0	1	*	0	1	*		
ψ	α	0	*	*	*	0	0	*
		1	*	*	*	0	1	*
		*	*	*	*	*	*	*
	β	0	*	*	*	*	*	*
		1	*	*	*	*	*	*
		*	*	*	*	*	*	*

βα-conjunction $\wedge[(x, \phi), (y, \psi)]$ is recorded as $\wedge(x, y)$ in brief. From the presented above it follows that for completely defined conditions (for $\phi = \beta$ and $\psi = \alpha$ and the truth values 0 and 1 of the logical variables x and y) *βα*-conjunction <x, y> has the same logical truth values as a classical conjunction [1]. However, *βα*-conjunction $\wedge(x, y)$ does not follow the law of contradiction $\wedge(x, \neg x) == 0$ and, besides, it sets the elementary position β to the logical variable x as well as it sets the elementary position α to the logical variable y.

Definition 5. The representation $\vee[(x, \phi), (y, \psi)] : P \times P \rightarrow B$ is the one that:

$$\vee[(x, \varphi), (y, \psi)] = \begin{cases} 1, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \psi = \alpha, \text{ if } x, y \in B, \\ \quad x = 1 \text{ and } y = 0, \text{ or } x = 0 \text{ and } y = 1, \text{ or } x = y = 1; \\ 0, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \psi = \alpha, \text{ if } x, y \in B \text{ and } x = y = 0; \\ *, \text{ in all other cases,} \end{cases}$$

and it is called *α-disjunction*.

The truth values of *α*-disjunction for logical variables x, y and the variables of the elementary positions ϕ, ψ are presented in Table 5.

Table 5

The truth values of α-disjunction ∨[(x, ϕ), (y, ψ)]

x / y			ϕ					
	α			β				
	0	1	*	0	1	*		
ψ	α	0	0	1	*	*	*	*
		1	1	1	*	*	*	*
		*	*	*	*	*	*	*
	β	0	*	*	*	*	*	*
		1	*	*	*	*	*	*
		*	*	*	*	*	*	*

α-disjunction $\wedge[(x, \phi), (y, \psi)]$ is recorded as $\vee(x, y)$ in brief.

From the presented above definition and the table, it follows that for completely defined conditions (for $\phi = \psi = \alpha$ and the truth values 0 and 1 of the logical variables x and y) α-disjunction $\vee(x, y)$ has the same logical truth values as a classical disjunction [1].

However, α-disjunction $\vee(x, y)$ does not follow the law of excluded middle $\vee(x, \neg x) == 1$ and, besides, it sets the position α to the logical variable x as well as it sets the elementary position α to the logical variable y.

Definition 6. The representation $[(x, \phi), (y, \psi)]\vee : P \times P \rightarrow B$ is the one that:

$$[(x, \varphi), (y, \psi)]\vee = \begin{cases} 1, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \psi = \beta, \text{ if } x, y \in B, \\ \qquad x = 1 \text{ and } y = 0, \text{ or } x = 0 \text{ and } y = 1, \text{ or } x = y = 1; \\ 0, \text{ for } \varphi, \psi \in \Pi \text{ and } \varphi = \psi = \beta, \text{ if } x, y \in B \text{ and } x = y = 0; \\ *, \text{ in all other cases,} \end{cases}$$

and it is called *β-disjunction*.

The truth values of β-disjunction for logical variables x, y and the variables of the elementary positions ϕ, ψ are presented in Table 6.

Table 6

The truth values of β-disjunction [(x, ϕ), (y, ψ)]∨

x / y			ϕ					
	α			β				
	0	1	*	0	1	*		
ψ	α	0	*	*	*	*	*	*
		1	*	*	*	*	*	*
		*	*	*	*	*	*	*
	β	0	*	*	*	0	1	*
		1	*	*	*	1	1	*
		*	*	*	*	*	*	*

β-disjunction $[(x, \phi), (y, \psi)]\vee$ is recorded as $(x, y)\vee$ in brief.

From the presented above, it follows that for completely defined conditions (for $\phi = \psi = \beta$ and the truth values 0 and 1 of the logical variables x and y) β-disjunction $(x, y)\vee$ has the same logical truth values as a classical disjunction [1].

However, β-disjunction $(x, y)\vee$ does not follow the law of excluded middle $\vee(x, \neg x) == 1$ and, besides, it sets the position β to the logical variable x as well as it sets the elementary position β to the logical variable y.

Definition 7. The representation $/(x, \phi), (y, \psi)/ : P \times P \rightarrow B$ is the one that:

$$/(x,\ \varphi),\ (y,\ \psi)/ = \begin{cases} 1, \text{ for } \varphi,\ \psi \in \Pi \text{ and } \varphi = \alpha,\ \psi = \beta, \text{ if } x, y \in B, \\ \qquad x = 1 \text{ and } y = 0, \text{ or } x = 0 \text{ and } y = 1, \text{ or } x = y = 1; \\ 0, \text{ for } \varphi,\ \psi \in \Pi \text{ and } \varphi = \alpha,\ \psi = \beta, \text{ if } x, y \in B \text{ and } x = y \\ *, \text{ in all other cases,} \end{cases}$$

$= 0;$

and it is called *αβ-disjunction*.

The truth values of $\alpha\beta$-disjunction for logical variables x, y and the variables of the elementary positions ϕ, ψ are presented in Table 7.

Table 7

The truth values of αβ-disjunction /(x, ϕ), (y, ψ)/

x / y			ϕ					
	α			β				
	0	1	*	0	1	*		
ψ	α	0	*	*	*	*	*	*
		1	*	*	*	*	*	*
		*	*	*	*	*	*	*
	β	0	0	1	*	*	*	*
		1	1	1	*	*	*	*
		*	*	*	*	*	*	*

In brief, $\alpha\beta$-disjunction $/(x, \phi), (y, \psi)/$ is recorded as $/x, y/$.

From the presented above, it follows that for completely defined conditions (for $\phi = \alpha$ and $\psi = \beta$ and the truth values 0 and 1 of the logical variables x and y) $\alpha\beta$-disjunction $/x, y/$ has the same logical truth values as a classical disjunction [1].

However, $\alpha\beta$-disjunction $/x, y/$ does not follow the law of excluded middle $/x, \neg x/ == 1$ and, besides, it sets the position α to the logical variable x and it sets the elementary position β to the logical variable y.

Definition 8. The representation $\backslash(x, \beta), (y, \alpha)\backslash : P \times P \rightarrow B$ is the one that:

$$\backslash(x,\ \varphi),\ (y,\ \psi)\backslash = \begin{cases} 1, \text{ for } \varphi,\ \psi \in \Pi \text{ and } \varphi = \beta,\ \psi = \alpha, \text{ if } x, y \in B, \\ \qquad x = 1 \text{ and } y = 0, \text{ or } x = 0 \text{ and } y = 1, \text{ or } x = y = 1; \\ 0, \text{ for } \varphi,\ \psi \in \Pi \text{ and } \varphi = \beta,\ \psi = \alpha, \text{ if } x, y \in B \text{ and } x = y \\ *, \text{ in all other cases,} \end{cases}$$

$= 0;$

and it is called *βα-disjunction*.

The truth values of $\beta\alpha$-disjunction for logical variables x, y and the variables of the elementary positions ϕ, ψ are presented in Table 8.

Table 8

The truth values of βα-disjunction \(x, ϕ), (y, ψ)

x / *y*			ϕ					
	α			β				
	0	1	*	0	1	*		
ψ	*α*	0	*	*	*	0	1	*
		1	*	*	*	1	1	*
		*	*	*	*	*	*	*
	β	0	*	*	*	*	*	*
		1	*	*	*	*	*	*
		*	*	*	*	*	*	*

$\beta\alpha$-disjunction \(x, β), (y, α)\ is recorded as \x, y\ in brief.

From the presented above, it follows that for completely defined conditions (for $\phi = \beta$ and $\psi = \alpha$ and the truth values 0 and 1 of the logical variables x and y) $\beta\alpha$-disjunction \x, y\ has the same logical truth values as a classical disjunction [1].

However, $\beta\alpha$-disjunction \$x$, y\ does not follow the law of excluded middle \$x$, $\neg x$\ == 1 and, besides, it sets the position β to the logical variable x and it sets the elementary position α to the logical variable y.

Definition 9. The representation ¬ is the one that:

$$\neg 1 = 0,$$

$$\neg 0 = 1,$$

$$\neg * = *,$$

$$\neg\#((x, \phi), (y, \psi)) = @((\neg x, \phi), (\neg y, \psi)),$$

if #[(x, ϕ), (y, ψ)] = &[(x, α), (y, α)], then @[($\neg x$, α), ($\neg y$, α)] = ∨[($\neg x$, ϕ), ($\neg y$, ψ)],

if #[(x, ϕ), (y, ψ)] = [(x, β), (y, β)]&, then @[($\neg x$, β), ($\neg y$, β)] = [($\neg x$, β), ($\neg y$, β)]∨,

if #[(x, ϕ), (y, ψ)] = <(x, α), (y, β)>, then @[($\neg x$, α), ($\neg y$, β)] = /($\neg x$, α), ($\neg y$, β)/,

if #[(x, ϕ), (y, ψ)] = ∧[(x, β), (y, α)], then @[($\neg x$, β), ($\neg y$, α)] = \($\neg x$, β), ($\neg y$, α)\,

if #[(x, ϕ), (y, ψ)] = ∨[(x, α), (y, α)], then @[($\neg x$, α), ($\neg y$, α)] = &[($\neg x$, α), ($\neg y$, α)],

if #[(x, ϕ), (y, ψ)] = [(x, β), (y, β)]∨, then @[($\neg x$, β), ($\neg y$, β)] = [($\neg x$, β), ($\neg y$, β)]&,

if #[(x, ϕ), (y, ψ)] = /(x, α), (y, β)/, then @[($\neg x$, α), ($\neg y$, β)] = <($\neg x$, α), ($\neg y$, β)>

and if #[(x, ϕ), (y, ψ)] = \(x, β), (y, α)\, then @[($\neg x$, β), ($\neg y$, α)] = <[($\neg x$, β), ($\neg y$, α)],

and it is called *positional inverting.*

Definition 10. Formulas or orderings are:

i) α-, β-, $\alpha\beta$-, $\beta\alpha$-conjunctions and disjunctions,

ii) if F is a formula, then $\neg F$ is a formula.

Any expression F is a formula if it can be shown using the items *i) – ii)* with a finite number of times.

Definition 11. The sequence of elementary positions of logical constants, variables and formulas formed by α-, β-, $\alpha\beta$-, $\beta\alpha$-conjunctions (disjunctions) is called *a position* and constants, variables and formulas with positions are called *positional.*

For example, $\alpha\beta$-conjunction and α- conjunction are present in the formula $<x, \&(y, z)>$. In α-conjunction $\&(y, z)$ the logical variables y and z have the elementary position α and α, i.e. we have pairs (y, α) and (z, α). α-conjunction itself in $\alpha\beta$-conjunction has the elementary position β, and the logical variable x has the elementary position α. Taking into account the elementary position β in $\alpha\beta$-conjunction for positional variables (y, α) and (z, α) leads to the positional variables $((y, \alpha), \beta)$ and $((z, \alpha), \beta)$. Thus, the positional variables x, y and z have the positions α, $\alpha\beta$ and $\alpha\beta$ in the formula $<x, \&(y, z)>$.

In formulas, as a rule, the positions with the positional variables are omitted to simplify the formulas.

3. Properties of α-, β-, $\alpha\beta$-, $\beta\alpha$-conjunctions and disjunctions

We will show that there is the equality $\&(x, 1) = x$ for $j = y = a$.

Table 9 presents the truth values of the formula $\&(x, 1)$ for the logical variable x and two variables of the positions j and y.

Table 9

The truth values of the formula &(x, 1)

x	j	1	y	$\&(x, 1)$
0	a	1	a	0
0	a	1	b	*
0	b	1	a	*
0	b	1	b	*
1	a	1	a	1
1	a	1	b	*
1	b	1	a	*
1	b	1	b	*
*	a	1	a	*
*	a	1	b	*
*	b	1	a	*
*	b	1	b	*

As it can be seen in the table, for $j = y = a$ the truth values of the formula $\&(x, 1)$ are the same as for the logical variable x. Thus, we get $\&(x, 1) = x$ for $j = y = a$.

Similarlψ, we set the equalities $(x, 1)\& = x$ (for $\varphi = \psi = \beta$), $<x, 1> = x$ (for $\varphi = a$ and $\psi = \beta$), $<1, x> = x$ (for $\varphi = a$ and $\psi = \beta$), $\wedge(x, 1) = x$ (for $\varphi = \beta$ and $\psi = a$), $\wedge(1, x) = x$ (for $\varphi = \beta$ and $\psi = a$) and $\vee(x, 0) = x$ (for $\varphi = \psi = a$), $(x, 0)\vee = x$ (for $\varphi = \psi = a$), $/x, 0/ = x$ (for $\varphi = \alpha$ and $\psi = \beta$), $/0, x/ = x$ (for $\varphi = a$ and $\psi = \beta$), $\backslash x, 0\backslash = x$ (for $\varphi = \beta$ and $\psi = a$), $\backslash 0, x\backslash = x$ (for $\varphi = \beta$ and $\psi = a$).

We will show that for $\varphi = \psi = a$:

$$\&(x, 0) = \begin{cases} 0, \text{ if } x = 0 \text{ or } x = 1; \\ *, \text{ if } x = *. \end{cases}$$

Table 10 presents the truth values of the formula &(x, 0) received on the basis of the definition of a-conjunction for the logical variable x, the constant 0 and two variables of the positions j and y for which $j = y = a$.

Table 10

The truth values of the formula &(x, 0)

x	j	0	y	&(x, 0)
0	a	0	a	0
0	a	0	b	*
0	b	0	a	*
0	b	0	b	*
1	a	0	a	0
1	a	0	b	*
1	b	0	a	*
1	b	0	b	*
*	a	0	a	*
*	a	0	b	*
*	b	0	a	*
*	b	0	b	*

The formula &(x, 0) for $\varphi = \psi = \alpha$ gets the truth values 0 for the truth values 0 and 1 of the logical variable x. For x = * it has the value *.

The formulas (x, 0)& (for $\varphi = \psi = b$), <x, 0> (for $\varphi = \alpha$ and $\psi = b$), <0, x> (for $\varphi = \alpha$ and $\psi = b$), $\wedge(x, 0)$ (for $\varphi = b$ and $\psi = a$) and $\wedge(0, x)$ (for $\varphi = b$ and $\psi = a$) have the similar truth values.

We will show that for $\varphi = \psi = a$:

$$\vee(x, 1) = \begin{cases} 1, \text{ if } x = 0 \text{ or } x = 1; \\ *, \text{ if } x = *. \end{cases}$$

On the basis of the definition of a-disjunction, we get Table 11 of the truth values.

Table 11

The truth values of the formula &(x, 0)

x	j	1	y	$\vee(x, 1)$
0	a	1	a	1
0	a	1	b	*
0	b	1	a	*
0	b	1	b	*
1	a	1	a	1
1	a	1	b	*
1	b	1	a	*
1	b	1	b	*
*	a	1	a	*
*	a	1	b	*
*	b	1	a	*
*	b	1	b	*

As it is seen in the table, for $\varphi = \psi = \alpha$ the formula $\vee(x, 1)$ for the truth values 0 and 1 of the logical variable x gets the truth value 1. For $x = *$ its truth value is *.

Similarly, we get the truth values of the formulas $\vee(1, x)$ (for $\varphi = \psi = a$), $(x, 1)\vee$ (for $\varphi = \psi = b$), $(1, x)\vee$ (for $\varphi = \psi = b$), $/x, 1/$ (for $\varphi = \alpha$ and $\psi = b$), $/1, x/$ (for $\varphi = \alpha$ and $\psi = b$), $\backslash x, 1\backslash$ (for $\varphi = b$ and $\psi = a$) and $\backslash 1, x\backslash$ (for $\varphi = b$ and $\psi = a$).

We will show that α-conjunction is commutative for $\varphi = \psi = \alpha$.

From the definition of α-conjunction, the formulas $\&(x, y)$ and $\&(y, x)$, for $\varphi = \psi = \alpha$, have the truth value 1 only when $x = 1$ and $\psi = 1$. These formulas get the truth value 0 only when x or ψ get the truth value 0. In all other cases, their truth value is *. So, the formulas $\&(x, y)$ and $\&(y, x$ have the same truth values) for all possible truth values of their variables. Thus, $\&(x, y) = \&(y, x)$.

Similarly, the formulas $(x, y)\&$, $\vee(x, y)$ and $(x, y)\vee$ are commutative.

We will show that a-conjunction is associative.

Associativity of a-conjunction means that $\&[\&(x, y), z] = \&[x, \&(y, z)]$.

As it is shown above, $x = \&(x, 1)$ and $z = \&(1, z)$. In the formula $\&[\&(x, y), z]$ we will replace the positional logical variable z (with the position a) with the formula $\&(1, z)$. Similarly, in the formula $\&[x, \&(y, z)]$ we will replace the positional variable x (with the position a) with $\&(x, 1)$.

We will get this equality as α result of the performed changes:

$$\&[\&(x, y), \&(1, z)] = \&[\&(x, 1), \&(y, z)].$$

All its variables have the same position aa. As it follows from the definition of a-conjunction, it gets the truth value 1 if and only if its two positional variables simultaneously have the truth values 1. If $x = y = z = 1$, then obviously the left and right sides of the equality will have the truth values 1. a-conjunction has the truth value 0 if and only if one or both of its positional variables get the truth value 0. If one assumes that one of the positional variables x, y and z, or any two or all at the same time, have the truth values 0, then the left and right sides of the equality will also have the truth value 0. In all other cases, the left and right sides of the equality will have the value *. Thus, on the basis of the same truth values of both formulas, with all possible combinations of the truth values of their variables, the associativity of a-conjunction has been proved.

Similarly, for $j = y = b$, one can define the associativity of b-conjunction:

$$[(x, y)\&, z]\& = [x, (y, z)\&]\&,$$

As well as a- and b-disjunction:

$$\vee[\vee(x, y), z] = \vee[x, \vee(y, z)],$$

$$[(x, y)\vee, z]\vee = [x, (y, z)\vee]\vee.$$

Table 12 presents that $<x, y> = \wedge(y, x)$ and $/x, y/ = \backslash y, x\backslash$.

Table 12

The truth values of the formulas <x, y>, ∧(y, x), /x, y/ and \y, x

x	j	y	*y*	<*x*, *y*>	∧(*y*, *x*)	/*x*, *y*/	*y*, *x*\\
0	*a*	0	*a*	*	*	*	*
0	*a*	0	*b*	0	0	0	0
0	*a*	1	*a*	*	*	*	*
0	*a*	1	*b*	0	0	1	1
0	*a*	*	*a*	*	*	*	*
0	*a*	*	*b*	*	*	*	*
0	*b*	0	*a*	*	*	*	*
0	*b*	0	*b*	*	*	*	*
0	*b*	1	*a*	*	*	*	*
0	*b*	1	*b*	*	*	*	*
0	*b*	*	*a*	*	*	*	*
0	*b*	*	*b*	*	*	*	*
1	*a*	0	*a*	*	*	*	*
1	*a*	0	*b*	0	0	1	1
1	*a*	1	*a*	*	*	*	*
1	*a*	1	*b*	1	1	1	1
1	*a*	*	*a*	*	*	*	*
1	*a*	*	*b*	*	*	*	*
1	*b*	0	*a*	*	*	*	*
1	*b*	0	*b*	*	*	*	*
1	*b*	1	*a*	*	*	*	*
1	*b*	1	*b*	*	*	*	*
1	*b*	*	*a*	*	*	*	*
1	*b*	*	*b*	*	*	*	*
*	*a*	0	*a*	*	*	*	*
*	*a*	0	*b*	*	*	*	*
*	*a*	1	*a*	*	*	*	*
*	*a*	1	*b*	*	*	*	*
*	*a*	*	*a*	*	*	*	*
*	*a*	*	*b*	*	*	*	*
*	*b*	0	*a*	*	*	*	*
*	*b*	0	*b*	*	*	*	*
*	*b*	1	*a*	*	*	*	*
*	*b*	1	*b*	*	*	*	*
*	*b*	*	*a*	*	*	*	*
*	*b*	*	*b*	*	*	*	*

a-conjunction is idempotent, which is shown in Table 13. *b*-conjunction, *a*- and *b*-disjunction are also idempotent.

We will define that there is the equality ∨[&(*x*, *y*), <*x*, *y*>] = &(*x*, /*y*, *y*/). The formulas ∨[&(*x*, *y*), <*x*, *y*>] and &(*x*, /*y*, *y*/) with clearly recorded positions of the logical variables look the following way: ∨[&[(*x*, *aa*), (*y*, *aa*)], <(*x*, *aa*), (*y*, *ba*)>] and &[(*x*, *a*), /(*y*, *aa*), (*y*, *ba*)/]. From the formula &[(*x*, *a*), /(*y*, *aa*), (*y*, *ba*)/] replacing the positional logical variable x (with the position a) with $\&(x, 1)$, we get $\&[\&[(x, aa), (1, a)],\ /(y, aa), (y, ba)/]$. Both in the formula $\vee[\&[(x, aa), (y, aa)], <(x, aa), (y, ba)>]$ and in the formula $\&\{\&[(x, aa), (1, a)],\ /(y, aa), (y, ba)/\}$ the variable x has the position aa, and the variable y has the positions aa and ba. These formulas have the truth values 1 only when the variable x is true and the variable y in the position aa or in the position ba or simultaneously in both positions gets the value 1. The formulas get the truth value 0 only when $x = 0$ and y in the position aa or in the position ba or simultaneously in both positions gets the value 0. In all other cases the formulas $\vee[\&[(x, aa), (y, aa)], <(x, aa), (y, ba)>]$ and $\&\{\&[(x, aa), (1, a)],\ /(y, aa), (y, ba)/\}$ get the value *. Thus, with all possible truth values of the positional variables of these formulas, there is the equality $\vee[\&(x, y), <x, y>] = \&(x, /y, y/)$.

Table 13

The truth values of the formula &(x, x)

x	j	y	$\&(x, x)$
0	a	a	0
0	a	b	*
0	b	a	*
0	b	b	*
1	a	a	1
1	a	b	*
1	b	a	*
1	b	b	*
*	a	a	*
*	a	b	*
*	b	a	*
*	b	b	*

Similarly, we can define that there are the equalities:

$$\vee[\&(x, y), \&\,(x, z)] = \&[x, \vee(y, z)],$$

$$[(x, y)\&, (x, z)\&]\vee = [x, (y, z)\vee]\&,$$

$$(<x, y>, <x, z>)\vee = <x, (y, z)\vee>,$$

$$\vee(<x, y>, <x, z>) = <x, \vee(y, z)>,$$

$$(<x, z>, <y, z>)\vee = <(x, y)\vee, z)>,$$

$$\vee(<x, z>, <y, z>) = <\vee(x, y), z>.$$

We will define if there are the equalities $\varnothing<x, y> = /\varnothing x, \varnothing y/$ and $\varnothing/x, y/ = <\varnothing x, \varnothing y>$. Table 14 presents the truth values of the formulas $\varnothing<x, y>$, $/\varnothing x, \varnothing y/$, $\varnothing/x, y/$ and $<\varnothing x, \varnothing y>$.

Table 14

The truth values of the formulas Ø<x, y>, /Øx, Øy/, Ø/x, y/ and <Øx, Øy>

x	j	y	y	<x, y>	¬<x, y>	/¬x, ¬y/	/x, y/	¬/x, y/	<¬x, ¬y>
0	1	2	3	4	5	6	7	8	9
0	a	0	a	*	*	*	*	*	*
0	a	0	b	0	1	1	0	1	1
0	a	1	a	*	*	*	*	*	*
0	a	1	b	0	1	1	1	0	0
0	a	*	a	*	*	*	*	*	*
0	a	*	b	*	*	*	*	*	*
0	b	0	a	*	*	*	*	*	*
0	b	0	b	*	*	*	*	*	*
0	b	1	a	*	*	*	*	*	*
0	b	1	b	*	*	*	*	*	*
0	b	*	a	*	*	*	*	*	*
0	b	*	b	*	*	*	*	*	*
1	a	0	a	*	*	*	*	*	*
1	a	0	b	0	1	1	1	0	0
1	a	1	a	*	*	*	*	*	*
1	a	1	b	1	0	0	1	0	0
1	a	*	a	*	*	*	*	*	*
1	a	*	b	*	*	*	*	*	*
1	b	0	a	*	*	*	*	*	*
1	b	0	b	*	*	*	*	*	*
1	b	1	a	*	*	*	*	*	*
1	b	1	b	*	*	*	*	*	*
1	b	*	a	*	*	*	*	*	*
1	b	*	b	*	*	*	*	*	*
*	a	0	a	*	*	*	*	*	*
*	a	0	b	*	*	*	*	*	*
*	a	1	a	*	*	*	*	*	*
*	a	1	b	*	*	*	*	*	*
*	a	*	a	*	*	*	*	*	*
*	a	*	b	*	*	*	*	*	*
*	b	0	a	*	*	*	*	*	*
*	b	0	b	*	*	*	*	*	*
*	b	1	a	*	*	*	*	*	*
*	b	1	b	*	*	*	*	*	*
*	b	*	a	*	*	*	*	*	*
*	b	*	b	*	*	*	*	*	*

The truth values in the columns 5 and 6, as well as in the columns 8 i 9 of Table 14 are identical with all possible truth values of the positional variables. Considering this, we can state that there are the equalities Ø<x, y> = /Øx, Øy/ and Ø/x, y/ = <Øx, Øy>.

The example of applying the introduced means for the algorithm description. Figure 1 presents the well-known block diagram of the algorithm ([2] Fig. 3.2 (a)), which is formed by the operator vertices with the assignment $s \leftarrow 0$, multiplication and assignment $t \leftarrow 2 \times s$, addition and assignment $s \leftarrow s + 1$ and the conditional vertex with the relation $t > 10$.

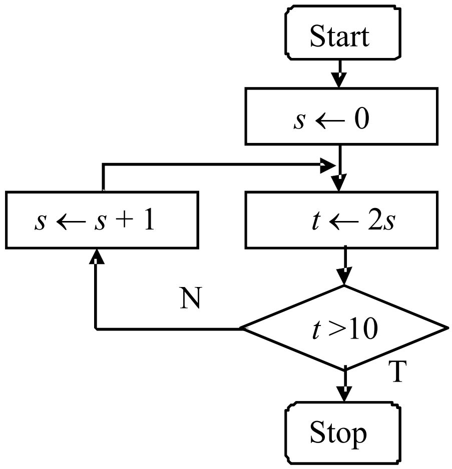

Fig. 1. Block diagram of the algorithm

In general, the assignment $s \leftarrow r$ is binary, which attributes the value of the variable r to the variable s. Its partial value is 0 in this example.

The assignment $s \leftarrow r$ corresponds to the predicate $\forall r \exists s\ (s \leftarrow r)$ as the one that:

$$\forall r\ \exists s\ (s \leftarrow r) = \begin{cases} 1, \text{ if } s = r; \\ 0, \text{ if } s \neq r. \end{cases}$$

After performing the assignment $s \leftarrow 0$ the variable s will have the value 0. The equality of the values s and 0 means that with this range of values (0 for s and the constant 0) the assignment $s \leftarrow 0$ is the predicate $0 \leftarrow 0$, which has the truth value 1. Thus, $s \leftarrow 0$ with the range of values 0 for s and 0, which are acquired after performing $s \leftarrow 0$, the assignment $0 \leftarrow 0$ is simultaneously the predicate with the truth value 1.

The expression $t \leftarrow 2 \times s$ is formed by multiplying 2 by s and the assignment of the result of multiplication to the variable t. The multiplication is binary and it is performed over s and k, the partial value of which is 2 in this example. We will record the multiplication as $k \times s$.

The multiplication $k \times s$ with the assignment $t \leftarrow k \times s$ of its result to the variable t corresponds to the predicate of multiplication $\forall k\ \forall s\ \exists t\ \ (t \leftarrow k \times s)$ as the one that:

$$\forall k\ \forall s\ \exists t\ (t \leftarrow k \cdot s) = \begin{cases} 1, \text{ if } t = k \cdot s; \\ 0, \text{ if } t \neq k \cdot s. \end{cases}$$

After performing the described by $t \leftarrow k \times s$ actions with the given specific range of values ($s = 0$, $k = 2$, $t = 2 \times s = 0$) $0 \leftarrow 2 \times 0$ is actually the predicate of multiplication which is true in this range of values.

The expression $s \leftarrow s + 1$ describes the binary addition and the assignment of this result. Similarly to the multiplication, we get the addition $s \leftarrow s + g$, which corresponds to the predicate $\forall s_{,}\ \forall g\ \exists s\ (s_{,} \leftarrow s + g)$ as the one that:

$$\forall s'\ \forall g\ \exists s\ (s' \leftarrow s + g) = \begin{cases} 1, \text{ if } s' = s + g; \\ \\ 0, \text{ if } s' \neq s + g, \end{cases}$$

for which the value s' is saved in s after performing the assignment. For $g = 1$ we get $s' \leftarrow s + 1$. After the performed by $s' \leftarrow s + 1$ actions with the given specific range of values ($s' = 1$, $s = 0, 1$) the assignment $1 \leftarrow 0 + 1$ is actually the predicate of addition which is true with this range of values.

The relation $t > 10$ is binary and it is true or false.

The block diagram of the algorithm presented in Figure 1 is described by *αβ*-conjunction:

$$<A, B>,$$

in which $A = s \leftarrow 0$,

$$B = <C, \vee[<\neg D, <E, B>>, D]>,$$

$C = t \leftarrow 2 \times s$, $D = (t > 10)$ and $E = s_{,} \leftarrow s + 1$.

In the extended form, the first iteration of the algorithm, if we interpret it over the specific range of values $s \leftarrow 0$, $t \leftarrow 2 \times s$, $t > 10$ and $s_{,} \leftarrow s + 1$ by predicates, is described by the expression:

$$<(s \leftarrow 0), <(t \leftarrow 2 \times s), \vee[<\neg(t > 10), <(s_{,} \leftarrow s + 1), B>>, (t > 10)]>>.$$

We will define its truth values. After performing the attribution of the value 0 to $s_{,}$ the assignment $0 \leftarrow 0$ is a true predicate. Considering this (on the basis that there is the equality $<1, x> = x$ for *ab*-conjunction), from the above formula we get:

$$<(t \leftarrow 2 \times s), \vee[<\neg(t > 10), <(s_{,} \leftarrow s + 1), B>>, (t > 10)]>$$

In the multiplication-predicate $t \leftarrow 2 \times s$ the value of the variable s is the value, which is acquired in the assignment-predicate $s \leftarrow 0$ and it equals 0. After performing the multiplication 2×0 the acquired value 0 is attributed to the variable t ($t = 0$). With the given range of values $t = 0, 2$ and $s = 0$ the multiplication-predicate $t \leftarrow 2 \times s$ has the truth value 1. Thus, taking this into consideration, from the last formula-ordering we get:

$$\vee[<\neg(t > 10), <(s \leftarrow s + 1), B>>, (t > 10)].$$

The relation $t > 10$ for the given value $t = 0$ gets the truth value 0. Thus, from the formula $\vee[<\neg(t > 10), <(s_{,} \leftarrow s + 1), B>>, (t > 10)]$ we have:

$$<(s_{,} \leftarrow s + 1), B>.$$

We calculate the value $s_{,} = 1$ with the help of the addition-predicate $s_{,} \leftarrow s + 1$, based on the value $s = 0$ and 1, which has been defined earlier. The truth value of this addition-predicate equals 1. Therefore, from the last formula we get B, which describes the transition to the second iteration.

The first and the second iterations are described by the formula:

$$<(s \leftarrow 0), <(t \leftarrow 2 \times s), \vee[<\neg(t > 10), <(s_{,} \leftarrow s + 1),$$

$$<(t \leftarrow 2 \times s), \vee[<\neg(t > 10), <(s_{,} \leftarrow s + 1), B>>, (t > 10)]>>>, (t > 10)]>>.$$

On the second iteration it starts with the multiplication-predicate $t \leftarrow 2 \times s$ with the range of values $s = 1$ (acquired by the addition-predicate $s_{,} \leftarrow s + 1$ on the first iteration), 2 and the calculated value $t = 2$. With this range of values, the multiplication-predicate $2 \leftarrow 2 \times 1$ is true. The relation $t > 10$ for the acquired $t = 2$ has the truth value 0. Thus, from the formula $\vee[<\neg(t > 10), <(s_{,} \leftarrow s + 1), B>>, (t > 10)]$ we get $<\neg(t > 10), <(s_{,} \leftarrow s + 1), B>> = <(s_{,} \leftarrow s + 1), B>$. After calculating the addition-predicate $(s_{,} \leftarrow s + 1)$, which with this range of values ($s_{,} = 3$, $s = 2$ and 1) is true, the next iteration starts with B.

4. Conclusions

The introduced α- and β-conjunctions $\&[(x, \alpha), (y, \alpha)]$ and $[(x, \beta), (y, \beta)]\&$ over $x, y \in \{0, 1\}$ have the same truth values as the classical conjunction $x \,\&\, y$. They are idempotent, commutative and associative. But they do not follow the law of contradiction ($\&[(x, \alpha), (\neg x, \alpha)] == 0$ and $[(x, \beta), (\neg x, \beta)]\& == 0$ for $x = *$). α-conjunction sets the elementary position α and β-conjunction sets the elementary position β to the logical variables x and y.

$\alpha\beta$- and $\beta\alpha$-conjunctions $<(x, \alpha), (y, \beta)>$ and $\wedge[(x, \beta), (y, \alpha)]$ over $x, y \in \{0, 1\}$ have the same truth values as the classical conjunction $x \,\&\, y$. But they are not idempotent, commutative and associative. They do not follow the law of contradiction ($<(x, \alpha), (\neg x, \alpha)> == 0$ and $\wedge[(x, \beta), (\neg x, \beta)] == 0$ for $x = *$). $\alpha\beta$-conjunction sets the elementary positions α and β, and $\beta\alpha$-conjunction sets the elementary positions β and α to the logical variables x and y.

The introduced α- and β-disjunctions $\vee[(x, \alpha), (y, \alpha)]$ and $[(x, \beta), (y, \beta)]\vee$ over $x, y \in \{0, 1\}$ have the same truth values as the classical disjunction $x \vee y$. They are idempotent, commutative and associative. But they do not follow the law of excluded middle ($\vee[(x, \alpha), (\neg x, \alpha)] == 1$ and $[(x, \beta), (\neg x, \beta)]\vee == 1$ for $x = *$). α-disjunction sets the elementary position α and β-disjunction sets the elementary position β to the logical variables x and y.

$\alpha\beta$- and $\beta\alpha$-disjunctions $/(x, \alpha), (y, \beta)/$ and $\backslash(x, \beta), (y, \alpha)\backslash$ over $x, y \in \{0, 1\}$ have the same truth values as the classical disjunction $x \vee y$. But they are not idempotent, commutative and associative. They do not follow the law of excluded middle ($/(x, \alpha), (\neg x, \alpha)/ == 1$ and $\backslash(x, \beta), (\neg x, \beta)\backslash == 1$ for $x = *$). $\alpha\beta$-disjunction sets the elementary positions α and β, and $\beta\alpha$-disjunction sets the elementary positions β and α to the logical variables x and y.

The positional inverting *α*-, *β*-, *αβ*- and *βα*-conjunctions (disjunctions) is performed exceptionally over their logical variables without any changes in their positions.

Positions set a unique ordering of positional constants, variables and formulas in the orderings formed by *α*-, *β*-, *αβ*-, *βα*-conjunctions (disjunctions) and the positional inverting.

Based on the properties of *α*-, *β*-, *αβ*-, *βα*-conjunctions (disjunctions) and the positional inverting, identical transformations are performed over the formulas-orderings.

The introduced *α*-, *β*-, *αβ*- and *βα*-conjunctions (disjunctions) and the positional inverting provide the calculation of the truth values of the formulas-orderings.

Due to the fact that for a specific range of values of variables and constants the operators of vertices of algorithms are simultaneously the predicates, then the algorithms must be described by *α*-, *β*-, *αβ*-, *βα*-conjunctions (disjunctions) and the positional inverting

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ОСНОВИ ПОЗИЦІЙНОЇ ЛОГІКИ

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В статті показано що логічні сталі і змінні не впорядковуються операціями класичної математичної логіки. Означеними в роботі α-, β-, αβ-, βα-кон'юнкціями (диз'юнкціями) і позиційним інвертуванням логічні константи, змінні і предикати впорядковуються. Наведено властивості α-, β-, αβ-, βα-кон'юнкцій (диз'юнкцій) і позиційного інвертування. Подано приклад застосування позиційної логіки для опису алгоритму.

Ключові слова: *α-кон'юнкція, β-кон'юнкція, αβ-кон'юнкція, βα-кон'юнкція, α-диз'юнкція, β-диз'юнкція, αβ-диз'юнкція, βα-диз'юнкція, позиційне інвертування.*

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